Zeroes of Polynomials

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Abstract- The problem of determining the best possible regions of all the zeroes of a polynomial with complex coefficients and in the complex variable is studied. Also the growth of the polynomial with restrictions on the zeroes is discussed.

Keywords- Polynomials, Analytic Functions, Rouchey's Theorem, Argument-principle, Lucas Theorem, Maximum Modulus Principle, Winding Number.

I. INTRODUCTION:

Let $p(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n$ be a polynomial of degree 'n' (n>=1) with complex coefficients a_i's and z is a complex variable.

By fundamental theorem of algebra p(z) vanishes at precisely 'n' points in the complex plane.

Say, at $z_1, z_2, \ldots\ldots, z_n$

Therefore,

$$p(z) = a_n \prod_{k=1}^{n}(z - z_k)$$

In this note, we shall be discussing mainly polynomial functions which are analytic in the complex plane or regions in the complex plane.

The basic results used are concerning analytic functions namely Rouchey's theorem, Argument principle, Lucas theorem, and Maximum Modulus Principle.

A complex number $\alpha \in \mathbb{C}$ is a zero of polynomial p(z),

$$\text{if } p(\alpha)=0, \sum_{k=0}^{n} a_k z^k = 0 \qquad (1)$$

we denote,

$$q(z) = z^n \overline{p(1/\bar{z})} \qquad (2)$$

as the conjugate polynomial to p(z), useful in studying the growth of the polynomial or its derivative with respect to its value on the unit disk.

We find

Therefore, $q(z) = z^k \bar{a}_n \prod_{k=1}^{n}(\frac{1}{z} - \bar{z}_k)$

$$= \bar{a}_n \prod_{k=1}^{n}(1 - z\bar{z}_k)$$

whose zeroes are $\frac{1}{\bar{z}k}$, $1 \le k \le n$

i.e. If $z_k = r_k e^{i\alpha_k}$, where r_k is the modulus of the zero and α_k is the argument

then the corresponding zero of q(z) is:

$1/\bar{z}_k = 1/\overline{r_k e^{i\alpha_k}}$

$= 1/ r_k e^{-i\alpha_k}$

$= e^{i\alpha_k}/r_k$

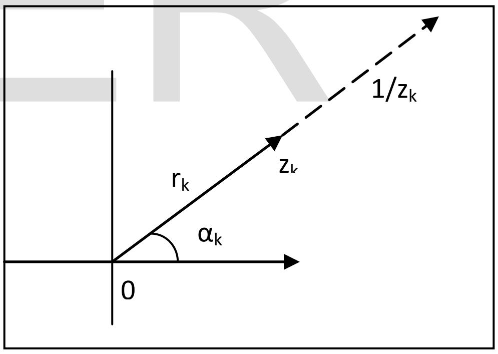

Fig.1. The figure shows the explanation of the problem

Therefore,

If the zero of p(z) lies inside the unit circle then corresponding zero of q(z) lies outside the unit circle and vice-versa.

i.e. if z_0 is a zero of p(z), its inverse point with respect to the unit disk is the zero of the corresponding polynomial q(z).

Hence, if all zeroes of p(z) lie inside the unit disk then all the zeroes of q(z) lie outside the unit disk whereas, a zero on the unit circle remains unaltered.

II. SECTION 1

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A sub class of polynomials is defined as p(z) = z^n p($\frac{1}{z}$) is called self-inversive.[1]

i.e. $a_j = a_{n-j}$ for j = 0,1, ... ,n.

If z_k is a zero of p(z) and p(z) is self inversive then $\frac{1}{zk}$ is also its zero.

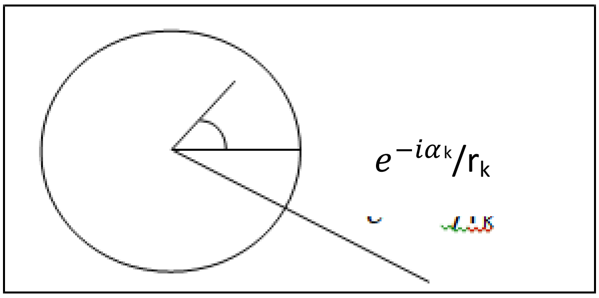

Fig.2. The fig. shows that If one root is inside the unit disk then other root has to be outside the unit disk.

Therefore,

$$z_k = r_k e^{i\alpha_k}$$

$$\text{then, } \frac{1}{z} = e^{-i\alpha_k}/r_k$$

i.e. If one root is inside the unit disk then other root has to be outside the unit disk as in fig.

Now, Let m(p, r) = max |p(z)|

Then, say : M = m(p,1)

We study the growth m(p, r) with respect to the M = m(p,1) as R varies

We know,

$$max_{|z|=R} \,|p(z)| \le R^n \, max_{|z|=1} \,|p(z)| \,,\ R \ge 1 \tag{3}$$

$$max_{|z|=1} \,|p'(z)| \le n \ max_{|z|=1} \,|p(z)| \tag{4}$$

In case, there is no zero in the unit disk, results (3), (4) have been improved by:

$$max_{|z|=R} \,|p(z)| \le (R^n+1)\, M/2 \ \text{(Ankene and Revelin)} \tag{5}$$

$$\text{And } max_{|z|=R} \,|p'(z)| \le n\, M/2 \ \text{(P.D. Lax)} \tag{6}$$

And both of them are best possible bounds attained when,

$$P(z) = (\alpha z^n+\beta)/2,\ |\alpha|=|\beta|=1$$

The inequality,

$$|p(Re^{i\theta})|+|q(Re^{i\theta})| \le (R^n + 1)\, M \tag{7}$$

Is responsible for bound of (5),

Now $p(Re^{i\theta})/\,q(Re^{i\theta})$ is analytic in |z|>1 when p(z) ≠ 0 in |z|<1.

On |z|=1,

$$|q(z)| = |\bar{a}_n \prod_{k=1}^{n}(1 - z\bar{z}_k)\,|$$

$$= |a_n| \prod_{k=1}^{n} |(1 - z\bar{z}_k)|$$

$$= |a_n| \prod_{k=1}^{n} |(z\bar{z} - z\bar{z}_k)| \quad \text{as } z\bar{z} = 1$$

$$= |a_n| \prod_{k=1}^{n} (|z||\bar{z} - \bar{z}_k)| \quad \text{as } |z| = 1$$

$$= |a_n| \prod_{k=1}^{n} |(z\bar{z} - z\bar{z}_k)|$$

$$= |a_n| \prod_{k=1}^{n} |(z - z_k)|$$

$$= |p(z)| \text{ when } |z|=1$$

Therefore, $\left|\frac{p(z)}{q(z)}\right| = 1$

By, Maximum- Modulus principle,

$$\left|\frac{p(z)}{q(z)}\right| \le 1 \text{ for } |z|>1$$

⇨ |p(z)| ≤ |q(z)| for |z|>1 and hence, when p(z) ≠ 0 in |z|<1 using (7)

We arrive at (5) i.e. the growth of $|p(Re^{i\theta})|$is determined with respect to M.

Next, we know all zero of p(z) = $a_0+a_1z+ \ldots\ldots +a_nz^n$

lie in $|z| \le 1+\max\left|\frac{aj}{an}\right|$ where 0 ≤ j ≤ n-1

say, = 1+A, where $\max\left|\frac{aj}{an}\right|$=A (Cauchy's theorem)

∴ The following result gives a better estimation of the region containing all zeroes with restrictions on the coefficients.

A. Further explanation

Consider p(z) = $a_0+a_1z+ \ldots\ldots +a_nz^n$, $a_n \ne 0$ with $|a_n|>|a_j|$; j=0, ... , n-1 then all the zeroes [3] of the p(z) lie in |z|≤ 2,

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Let $|p(z)| = |a_n z^n + \ldots + a_1 z + a_0| \geq |a_n z^n| - |a_{n-1} z^{n-1} + \ldots + a_1 z + a_0|$

$\geq |a_n||z^n| - (|a_n z^{n-1}| + \ldots + |a_0|)$

$= |a_n||z^n|\{1 - [(|a_{n-1}/a_n|)\frac{1}{|z|} + (|a_{n-2}/a_n|)\frac{1}{|z|^2} + \ldots + |a_0/a_n|\frac{1}{z^n}]\}$

$\geq |a_n||z^n|[1 - \{\frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{z^n}\}]$ as $|a_j| \leq |a_n|$

$= |a_n||z^n|(1 - \sum_{k=1}^{n} \frac{1}{|z|^k}) > |a_n||z^n|(1 - \sum_{k=1}^{\infty} \frac{1}{|z|^k})$

$= |a_n||z^n|(1 - \sum_{k=1}^{\infty} \frac{1}{|z|^k} + 1 - 1)$

$= |a_n||z^n|[2 - (1 + \sum_{k=1}^{n} \frac{1}{|z|^k})]$

$= |a_n||z^n|[2 - \{\frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{z^n} + \ldots + \infty\}]$

– as G.P. formed $\frac{1}{|z|} < 1$,

$= |a_n||z^n|[2 - (\frac{1}{1 - \frac{1}{|z|}})]$

$= |a_n||z^n|[2 - \frac{|z|}{|z|-1}]$

$= |a_n||z^n|(\frac{|z|-2}{|z|-1}) > 0$ when $|z|>2$

i.e. $|p(z)| > 0$ for all z, $|z|>2$

Hence, all the zeroes of p(z) lie in $|z| \leq 2$

Now,

We state another interesting well known result about the region containing all zeroes of a polynomial with more restrictions on the coefficients known as Enesterom-Kakeya Theorem

"If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n with real coefficients $0 \leq a_0 \ldots \leq a_n$ then all the zeroes of p(z) lie in $|z| \leq 1$".

III. SECTION 2

We state some well known results:

*2.1 **Rouchey's Theorem**:*

If p(z), q(z) are analytic interior to a simple closed Jordan Curve C and if they are continuous on C and $|p(z)| < |q(z)|$ on C,

Then, the f(z) = p(z) + q(z) has the same no. Of zeroes interior to C as does q(z).

*2.2. **Lucas Theorem**:*

Any convex polygon which contains all the zeroes of a p(z) also contains all the zeroes of its derivative.

In particular, any circle C which encloses all zeroes of p(z) also encloses all zeroes of p'(z).

*2.3. **Maximum Modulus Principle**:*

If G is a region and f: G→ C is an analytic function such that there is a point a in G with $|f(a)| \geq |f(z)|$ for all z in G then f is constant.

*2.4. **Winding Number**:*

"Let r be a closed rectifiable curve and a ∈ C but a ∉ γ

Then, $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer and is denoted by n(γ, a) called winding number"[2]

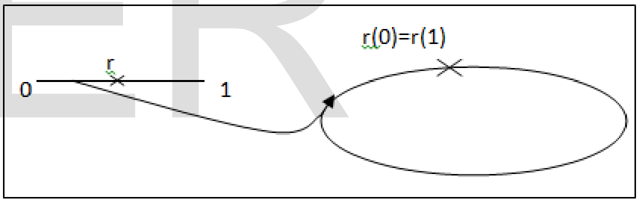

Fig.3. The figure provides the explanation for winding number problem.

proof:

Let z= r(s), s ∈ [0,1] (1)

Define, $\int_0^t \frac{r'(s)}{r(s)} ds = g(t)$ (2)

r is complex valued function of a real variable s

∴ g is derivable with r(0) = r(1) in [0, 1]

z= r(s)

dz = r(s)ds

$$\int_\gamma \frac{dz}{z-a} = g(1)$$

$$= \int_0^1 \frac{r'(s)}{r(s)-a} ds \qquad (3)$$

From (2) on differentiating with respect to t

by Leibnitz rule,

$g'(t) = \frac{d}{dt}\int_0^t \frac{r'(s)}{r(s)-a} ds = 0 + r'(t)/r(t)\text{-}a + 0$, $\gamma :[0, 1]\rightarrow C$

(4)

Now, $\frac{d}{dt}(e^{-g(t)}(r(t)-a))$

$= e^{-g(t)}\, r'(t) + (r(t)-a)\,(e^{-g(t)}(-g'(t)))$

$= e^{-g(t)}[r'(t) - g'(t)\,(r(t)-a)]$

$= e^{-g(t)}\,[r'(t) - \frac{r'(t)}{(r(t)-a)} \times (r(t)-a)]$

$= 0$ using (4) $[\therefore r(t) \neq a]$

Hence, $e^{-g(t)}(r(t)-a)$ is a constant function

In particular, at t=0, t=1;

$\therefore e^{-g(0)}(r(0)-a)$ is same constant

$= e^{-g(t)}\,(r(t)-a) = e^{-g(0)}(r(0)-a) = e^{-g(1)}(r(1)-a)$

as r(0) = r(1) as r(t) is closed

We get, $e^{-g(0)} = e^{-g(1)}$

But from (2), g(0)=0,

$\therefore e^{-g(0)} = e^0 = 1 = e^{-g(1)}$

But it is possible when $g(1) = 2\pi ik$ for k any integer $[e^{2\pi ik} = \cos 2\pi k + i\sin 2\pi k]$

Hence, $g(1) = 2\,\pi ik$

But from (3),

$$g(1) = \int_\gamma \frac{dz}{z-a} = 2\pi ik$$

$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ = k as some integer and where k becomes the Winding Number.

Now, results can be proved using this:

A. Explanation with example

Show any polynomial can be written in power of z-a and a is a complex no. Hence, without Cauchy's Integral Formula prove: $p(a) = \frac{1}{2\pi i}\int_\gamma \frac{p(z)dz}{z-a}$

Where, γ is a circle with center z=a

Proof: Let $p(z) = \sum_{k=0}^{n} a_k z^k$ (1)

$= \sum_{k=0}^{n} a_k (z-a+a)^k$

$= \sum_{k=0}^{n} a_k [\,{}^kC_0 (z-a)^k a^0 + \ldots\ldots + {}^kC_k a^k\,]$

Using Binomial Theorem,

It is repeating itself in degree of (z-a)

$p(z) = \sum_{j=0}^{n} A_j (z-a)^j$ (2)

$= A_0 + A_1(z-a) + \ldots + A_n(z-a)^n$

$p(a) = A_0 = \sum_{k=0}^{n} a_k z^k$ (3)

Now, $\frac{1}{2\pi i}\int_\gamma \frac{p(z)dz}{z-a}$

$= \frac{1}{2\pi i}\int_\gamma \sum_{j=0}^{n} Aj \frac{(z-a)^j dz}{z-a}$ using (2)

$= \frac{1}{2\pi i}\int_\gamma A_0/(z-a) + A_1 + A_2(z-a) + \ldots + A_n (z-a)^{n-1} dz$

$= \frac{1}{2\pi i}\int_\gamma A_0/(z-a)\, dz + 0 + \ldots + 0$

$= \frac{1}{2\pi i} Ao \int_\gamma dz/(z-a)$ $[\ldots \int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{e^{i\theta}\, id\theta}{e^{i\theta}} = 2\pi i]$

$= \frac{1}{2\pi i} Ao \times 2\pi i \times 1$

$= A_0 = p(a)$ by using (3). Hence, proved.

B, Another example

Locate the regions containing all the roots of equations:

$Z^5 + 15z + 1 = 0$

Consider, f(z) = 15z

$g(z) = z^5 + 1$

use notation zeroes of f(z)at |z| origin $|z| \leq 3/2$ at

$|f(z)| = 15 \times 3/2 = 45/2$

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Now, $|g(z)| = |z^5+1|$

$\leq |z^5| + 1$

$\leq |3/2|^5 + 1 = (243/32) + 1 = 245/2$

Therefore, $|g(z)| \leq |f(z)|$

Then, f(z) has only one zero

$\rightarrow |z| \leq 3/2$

Therefore, $g(z) + f(z)$ also has one zero in $|z| \leq 3/2$

Again consider, $f(z) = z^5$

$g(z) = 15z + 1$

$f|z| = 2^5 = 32$

$g|z| = 15x2+1 = 31$

thus, $g|z| < f|z|$; $|z|=2$

Also, f(z) containing all 5 zeroes in equation ≤ 2

Therefore, $g(z) + f(z) = z^5 + 15z+1$

Has all 5 zeroes in $|z| <=2$ in the given eq. Having zero in $|z|<=3/2$ and other four remaining $3/2 <=|z| <=2$.

C. Another example

Consider no. Of zeroes in $z^8 -4z^5+z^2-1 = 0$ lie in unit disk.

Denoting $f(z) = -4z^5$ and $g(z) = z^8+z^2-1$,

$|f(z)| = 4$ where eq. $|g(z)|<= 1+1+1 = 3$

Therefore, $|g(z)| < |f(z)|$ on $|z| =1$ now all 5 zeroes.

By Rouchey's theorem,

eq. $g(z) + f(z) = z^8 - 4z^4 + z^2 - 1$ lie in $|z| <=1$

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